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Effective Hamiltonian for $B \to X_s e^+ e^-$ Beyond Leading Logarithms in the NDR and HV Schemes *

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Abstract

We calculate the next-to-leading QCD corrections to the effective Hamiltonian for $B \to X_s e^+ e^-$ in the NDR and HV schemes. We give for the first time analytic expressions for the Wilson Coefficient of the operator $Q_9 = (\bar{s}b)_{V-A}(\bar{e}e)_V$ in the NDR and HV schemes. Calculating the relevant matrix elements of local operators in the spectator model we demonstrate the scheme independence of the resulting short distance contribution to the physical amplitude. Keeping consistently only leading and next-to-leading terms, we find an analytic formula for the differential dilepton invariant mass distribution in the spectator model. Numerical analysis of the $m_{\rm t}$, $\Lambda_{\overline{\rm MS}}$ and $\mu \approx \mathcal{O}(m_{\rm b})$ dependences of this formula is presented. We compare our results with those given in the literature.

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1 Introduction

The rare decay $B \to X_s e^+e^-$ has been the subject of many theoretical studies in the framework of the standard model and its extensions such as the two Higgs doublet models and models involving supersymmetry [1–8]. In particular the strong dependence of $B \to X_s e^+e^-$ on $m_{\rm t}$ has been stressed by Hou et. al. [1]. It is clear that once $B \to X_s e^+e^-$ has been observed, it will offer an useful test of the standard model and of its extensions. To this end the relevant branching ratio, the dilepton invariant mass distribution and other distributions of interest should be calculated with sufficient precision. In particular the QCD effects should be properly taken into account.

The central element in any analysis of $B \to X_s e^+e^-$ is the effective Hamiltonian for $\Delta B = 1$ decays relevant for scales $\mu \approx \mathcal{O}(m_{\rm b})$ in which the short distance QCD effects are taken into account in the framework of a renormalization group improved perturbation theory. These short distance QCD effects have been calculated over the last years with increasing precision by several groups [2, 9, 10] culminating in a complete next-to-leading QCD calculation presented by Misiak in ref. [11] and very recently in a corrected version in [12].

The actual calculation of $B \to X_s e^+e^-$ involves not only the evaluation of Wilson coefficients of ten local operators (see (2.1)) which mix under renormalization but also the calculation of the corresponding matrix elements of these operators relevant for $B \to X_s e^+e^-$. The latter part of the analysis can be done in the spectator model, which, according to heavy quark effective theory, for B-decays should offer a good approximation to QCD. One can also include the non-perturbative $\mathcal{O}(1/m_{\rm b}^2)$ corrections to the spectator model which enhance the rate for $B \to X_s e^+e^-$ by roughly 10% [13]. A realistic phenomenological analysis should also include the long distance contributions which are mainly due to the J/ψ and ψ' resonances [14–16]. Since in this paper we are mainly interested in the next-to-leading short distance QCD corrections to the spectator model we will not include these complications in what follows.

It is well known that the Wilson coefficients of local operators depend beyond the leading logarithmic approximation on the renormalization scheme for operators, in particular on the treatment of γ_5 in $D \neq 4$ dimensions. This dependence must be cancelled by the scheme dependence present in the matrix elements of operators so that the final decay amplitude does not depend on the renormalization scheme. In the context of $B \to X_s e^+e^-$ this point has been emphasized in particular by Grinstein et. al. [2]. Other examples such as $K \to \pi\pi$, $K_{L,S} \to \pi^0 e^+e^-$, $B \to X_s\gamma$ can be found in refs. [17–19]. The interesting feature of $B \to X_s e^+e^-$ as compared to decays such as $K \to \pi\pi$, is the fact that due to the ability of calculating reliably the matrix elements of all operators contributing to this decay, the cancellation of scheme dependence can be demonstrated in the actual calculation of the short distance part of the physical amplitude.

Now all the existing calculations of $B \to X_s e^+e^-$ use the NDR renormalization scheme (anticommuting γ_5 in $D \neq 4$ dimensions). Even if arguments have been given, in particular in [2] and [11], how the cancellation of the scheme dependence in $B \to X_s e^+e^-$ would take place, it is of interest to see this explicitly by calculating this decay in two different renormalization schemes. In addition, in view of the complexity of next-to-leading order (NLO) calculations and the fact that the only complete NLO analysis of $B \to X_s e^+e^-$ has

been done by a single person, it is important to check the results of refs. [11, 12].

Here we will present the calculations of the Wilson coefficients and matrix elements relevant for $B \to X_s e^+ e^-$ in two renormalization schemes (NDR and HV [20]) demonstrating the scheme independence of the resulting amplitude. Beside this the main results of our paper are as follows:

- We give for the first time analytic NLO expressions for the Wilson coefficient of the operator $Q_9 = (\bar{s}b)_{V-A}(\bar{e}e)_V$ in the NDR and HV schemes.

- Calculating the matrix elements of local operators in the spectator model we fully agree with Misiak's result for the dilepton invariant mass distribution very recently given in [12].

- We find, that in the HV scheme the scheme dependent term in the matrix elements (the so called ξ-term) receives in addition to current-current contributions also contributions from QCD penguin operators which are necessary for the cancellation of the scheme dependence in the final amplitude. This should be compared with the discussion of the scheme dependence given in refs. [2] and [11] where the ξ-term received only contributions from current-current operators.

- We stress that in a consistent NLO analysis of the decay $B \to X_s e^+ e^-$, one should on one hand calculate the Wilson coefficient of the operator $Q_9 = (\bar{s}b)_{V-A}(\bar{e}e)_V$ including leading and next-to-leading logarithms, but on the other hand only leading logarithms should be kept in the remaining Wilson coefficients. Only then a scheme independent amplitude can be obtained. This special treatment of Q_9 is related to the fact that strictly speaking in the leading logarithmic approximation only this operator contributes to $B \to X_s e^+ e^-$. The contributions of the usual current-current operators, QCD penguin operators, magnetic penguin operators and of $Q_{10} = (\bar{s}b)_{V-A}(\bar{e}e)_A$ enter only at the NLO level and to be consistent only the leading contributions to the corresponding Wilson coefficients should be included. In this respect we differ from the original analysis of Misiak [11] who in his numerical evaluation of $B \to X_s e^+ e^-$ also included partially known NLO corrections to Wilson coefficients of operators $Q_i (i \neq 9)$. These additional corrections are, however, scheme dependent and are really a part of still higher order in the renormalization group improved perturbation theory. The most recent analysis of Misiak [12] does not include these contributions and can be directly compared with the present paper.

- Keeping consistently only the leading and next-to-leading contributions to $B \to X_s e^+ e^-$ we are able to give analytic expressions for *all* Wilson coefficients which should be useful for phenomenological applications.

Our paper is organized as follows:

In sect. 2 we collect the master formulae for $B \to X_s e^+ e^-$ in the spectator model which include consistently leading and next-to-leading logrithms. In sect. 3 we describe some details of the NLO calculation of the Wilson coefficient $C_9(\mu)$ and of the relevant one-loop matrix elements in NDR and HV schemes. In sect. 4 we present a numerical analysis. We end our paper with a brief summary of the main results.

2 Master Formulae

2.1 Operators

Our basis of operators is given as follows:

$$
\begin{aligned}
Q_1 &= (\bar{s}_\alpha c_\beta)_{V-A}(\bar{c}_\beta b_\alpha)_{V-A} \\
Q_2 &= (\bar{s}c)_{V-A}(\bar{c}b)_{V-A} \\
Q_3 &= (\bar{s}b)_{V-A}\sum_q(\bar{q}q)_{V-A} \\
Q_4 &= (\bar{s}_\alpha b_\beta)_{V-A}\sum_q(\bar{q}_\beta q_\alpha)_{V-A} \\
Q_5 &= (\bar{s}b)_{V-A}\sum_q(\bar{q}q)_{V+A} \\
Q_6 &= (\bar{s}_\alpha b_\beta)_{V-A}\sum_q(\bar{q}_\beta q_\alpha)_{V+A} \\
Q_7 &= \tfrac{e}{8\pi^2}m_b\bar{s}_\alpha\sigma^{\mu\nu}(1+\gamma_5)b_\alpha F_{\mu\nu} \\
Q_8 &= \tfrac{g}{8\pi^2}m_b\bar{s}_\alpha\sigma^{\mu\nu}(1+\gamma_5)T^a_{\alpha\beta}b_\beta G^a_{\mu\nu} \\
Q_9 &= (\bar{s}b)_{V-A}(\bar{e}e)_V \\
Q_{10} &= (\bar{s}b)_{V-A}(\bar{e}e)_A
\end{aligned}
\tag{2.1}
$$

where α and β denote colour indices. We omit the colour indices for the colour-singlet currents. Labels $(V \pm A)$ refer to $\gamma_\mu(1\pm\gamma_5)$. $Q_{1,2}$ are the current-current operators, Q_{3-6} the QCD penguin operators, $Q_{7,8}$ "magnetic penguin" operators and $Q_{9,10}$ semi-leptonic electroweak penguin operators. Our normalizations are as in refs. [18] and [19].

2.2 Wilson Coefficients

The Wilson coefficients for the operators Q_1–Q_7 are given in the leading logarithmic approximation by [18, 21–23]

$$C_j^{(0)}(\mu) = \sum_{i=1}^{8} k_{ji}\eta^{a_i} \qquad (j = 1, ...6) \tag{2.2}$$

$$C_7^{(0)eff}(\mu) = \eta^{\frac{16}{23}}C_7^{(0)}(M_{\rm W}) + \frac{8}{3}\left(\eta^{\frac{14}{23}} - \eta^{\frac{16}{23}}\right)C_8^{(0)}(M_{\rm W}) + \sum_{i=1}^{8} h_i\eta^{a_i}, \tag{2.3}$$

with

$$\eta = \frac{\alpha_s(M_{\rm W})}{\alpha_s(\mu)}, \tag{2.4}$$

$$C_7^{(0)}(M_{\rm W}) = -\frac{1}{2}A(x_t), \tag{2.5}$$

$$C_8^{(0)}(M_{\rm W}) = -\frac{1}{2}F(x_t), \tag{2.6}$$

where $x_t = m_{\rm t}^2/M_{\rm W}^2$ and $A(x)$ and $F(x)$ are defined in (2.14) and (2.19). The numbers a_i, k_{ji} and h_i are given by

$$\begin{array}{rlrrrrrrrrl}
a_i = (& \frac{14}{23}, & \frac{16}{23}, & \frac{6}{23}, & -\frac{12}{23}, & 0.4086, & -0.4230, & -0.8994, & 0.1456 &) \\
k_{1i} = (& 0, & 0, & \frac{1}{2}, & -\frac{1}{2}, & 0, & 0, & 0, & 0 &) \\
k_{2i} = (& 0, & 0, & \frac{1}{2}, & \frac{1}{2}, & 0, & 0, & 0, & 0 &) \\
k_{3i} = (& 0, & 0, & -\frac{1}{14}, & \frac{1}{6}, & 0.0510, & -0.1403, & -0.0113, & 0.0054 &) \\
k_{4i} = (& 0, & 0, & -\frac{1}{14}, & -\frac{1}{6}, & 0.0984, & 0.1214, & 0.0156, & 0.0026 &) \\
k_{5i} = (& 0, & 0, & 0, & 0, & -0.0397, & 0.0117, & -0.0025, & 0.0304 &) \\
k_{6i} = (& 0, & 0, & 0, & 0, & 0.0335, & 0.0239, & -0.0462, & -0.0112 &) \\
h_i = (& 2.2996, & -1.0880, & -\frac{3}{7}, & -\frac{1}{14}, & -0.6494, & -0.0380, & -0.0186, & -0.0057 &).
\end{array} \tag{2.7}$$

The first correct calculation of the two-loop anomalous dimensions relevant for (2.3) has been presented in [21, 22] and confirmed subsequently in [12, 24, 25].

The coefficient $C_8^{(0)eff}(\mu)$ does not enter the formula for $B \to X_s e^+ e^-$ at this level of accuracy. An analytic formula is given in ref. [18].

The coefficient of Q_{10} is given by

$$C_{10}(M_{\rm W}) = \frac{\alpha}{2\pi}\widetilde{C}_{10}(M_{\rm W}), \qquad \widetilde{C}_{10}(M_{\rm W}) = -\frac{Y(x_t)}{\sin^2\Theta_W} \tag{2.8}$$

with $Y(x)$ given in (2.13). Since Q_{10} does not renormalize under QCD, its coefficient does not depend on $\mu \approx \mathcal{O}(m_{\rm b})$. The only renormalization scale dependence in (2.8) enters through the definition of the top quark mass. We will return to this issue in sect. 4.

Finally, including leading as well as next-to-leading logarithms, we find

$$C_9^{NDR}(\mu) = \frac{\alpha}{2\pi}\widetilde{C}_9^{NDR}(\mu) \tag{2.9}$$

$$\widetilde{C}_9^{NDR}(\mu) = P_0^{NDR} + \frac{Y(x_t)}{\sin^2\Theta_W} - 4Z(x_t) + P_E E(x_t) \tag{2.10}$$

with

$$P_0^{NDR} = \frac{\pi}{\alpha_s(M_W)}(-0.1875 + \sum_{i=1}^{8} p_i \eta^{a_i+1}) + 1.2468 + \sum_{i=1}^{8} \eta^{a_i}[r_i^{NDR} + s_i\eta] \tag{2.11}$$

$$P_E = 0.1405 + \sum_{i=1}^{8} q_i \eta^{a_i+1} \tag{2.12}$$

$$Y(x) = C(x) - B(x), \qquad Z(x) = C(x) + \frac{1}{4}D(x). \tag{2.13}$$

Here

$$A(x) = \frac{x(8x^2+5x-7)}{12(x-1)^3} + \frac{x^2(2-3x)}{2(x-1)^4}\ln x, \tag{2.14}$$

$$
\begin{aligned}
B(x) &= \frac{x}{4(1-x)} + \frac{x}{4(x-1)^2}\ln x, && (2.15)\\
C(x) &= \frac{x(x-6)}{8(x-1)} + \frac{x(3x+2)}{8(x-1)^2}\ln x, && (2.16)\\
D(x) &= \frac{-19x^3+25x^2}{36(x-1)^3} + \frac{x^2(5x^2-2x-6)}{18(x-1)^4}\ln x - \frac{4}{9}\ln x, && (2.17)\\
E(x) &= \frac{x(18-11x-x^2)}{12(1-x)^3} + \frac{x^2(15-16x+4x^2)}{6(1-x)^4}\ln x - \frac{2}{3}\ln x, && (2.18)\\
F(x) &= \frac{x(x^2-5x-2)}{4(x-1)^3} + \frac{3x^2}{2(x-1)^4}\ln x. && (2.19)
\end{aligned}
$$

The coefficients p_i, r_i^{NDR}, s_i, and q_i are found to be as follows:

$$
\begin{aligned}
p_i &= (\ 0,\ \ 0,\ \ -\tfrac{80}{203},\ \ \tfrac{8}{33},\ \ 0.0433,\ \ 0.1384,\ \ 0.1648\ \ -0.0073\ )\\
r_i^{NDR} &= (\ 0,\ \ 0,\ \ 0.8966,\ \ -0.1960,\ \ -0.2011,\ \ 0.1328,\ \ -0.0292,\ \ -0.1858\ )\\
s_i &= (\ 0,\ \ 0,\ \ -0.2009,\ \ -0.3579,\ \ 0.0490,\ \ -0.3616,\ \ -0.3554,\ \ 0.0072\ )\\
q_i &= (\ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0.0318,\ \ 0.0918,\ \ -0.2700,\ \ 0.0059\ ).
\end{aligned}
\tag{2.20}
$$

P_E is $\mathcal{O}(10^{-2})$ and consequently the last term in (2.10) can be neglected. We keep it however in our numerical analysis.

In the HV scheme only the coefficients r_i are changed. They are given by

$$
r_i^{HV} = (\ 0,\ \ 0,\ \ -0.1193,\ \ 0.1003,\ \ -0.0473,\ \ 0.2323,\ \ -0.0133,\ \ -0.1799\ ). \tag{2.21}
$$

Equivalently we can write

$$
P_0^{HV} = P_0^{NDR} + \xi^{HV}\frac{4}{9}\left(3C_1^{(0)} + C_2^{(0)} - C_3^{(0)} - 3C_4^{(0)}\right) \tag{2.22}
$$

with

$$
\xi = \left\{ \begin{array}{rl} 0, & \mathrm{NDR} \\ -1, & \mathrm{HV}. \end{array} \right. \tag{2.23}
$$

We note that

$$
\sum_{i=1}^{8} p_i = 0.1875, \qquad \sum_{i=1}^{8} q_i = -0.1405, \tag{2.24}
$$

$$
\sum_{i=1}^{8} (r_i + s_i) = -1.2468 + \frac{4}{9}(1+\xi), \qquad \sum_{i=1}^{8} p_i(a_i+1) = -\frac{16}{69}. \tag{2.25}
$$

In this way for $\eta = 1$ we find $P_E = 0$, $P_0^{NDR} = 4/9$ and $P_0^{HV} = 0$ in accordance with the initial conditions in (3.3). Moreover, the second relation in (2.25) assures the correct large logarithm in P_0^{NDR}, i. e. $8/9 \ln(M_{\rm W}/\mu)$. The derivation of (2.9)–(2.22) is given in sect. 3.

2.3 The Differential Decay Rate

Introducing

$$
\hat{s} = \frac{(p_{e^+} + p_{e^-})^2}{m_{\rm b}^2}, \qquad z = \frac{m_{\rm c}}{m_{\rm b}} \tag{2.26}
$$

and calculating the one-loop matrix elements of Q_i using the spectator model in the NDR scheme we find

$$
\begin{aligned}
R(\hat{s}) \equiv \frac{\frac{d}{d\hat{s}}\Gamma(b \to s e^+ e^-)}{\Gamma(b \to c e \bar{\nu})} &= \frac{\alpha^2}{4\pi^2}\left|\frac{V_{ts}}{V_{cb}}\right|^2 \frac{(1-\hat{s})^2}{f(z)\kappa(z)} \cdot \Big[(1+2\hat{s})\left(|\widetilde{C}_9^{eff}|^2 + |\widetilde{C}_{10}|^2\right) + \\
& 4\left(1+\frac{2}{\hat{s}}\right)|C_7^{(0)eff}|^2 + 12 C_7^{(0)eff}\ \mathrm{Re}\,\widetilde{C}_9^{eff}\Big]
\end{aligned} \tag{2.27}
$$

where

$$
\begin{aligned}
\widetilde{C}_9^{eff} &= \widetilde{C}_9^{NDR}\tilde{\eta}(\hat{s}) + h(z,\hat{s})\left(3C_1^{(0)} + C_2^{(0)} + 3C_3^{(0)} + C_4^{(0)} + 3C_5^{(0)} + C_6^{(0)}\right) \\
& -\frac{1}{2}h(1,\hat{s})\left(4C_3^{(0)} + 4C_4^{(0)} + 3C_5^{(0)} + C_6^{(0)}\right) \\
& -\frac{1}{2}h(0,\hat{s})\left(C_3^{(0)} + 3C_4^{(0)}\right) + \frac{2}{9}\left(3C_3^{(0)} + C_4^{(0)} + 3C_5^{(0)} + C_6^{(0)}\right).
\end{aligned} \tag{2.28}
$$

Here

$$
\begin{aligned}
h(z,\hat{s}) &= -\frac{8}{9}\ln\frac{m_b}{\mu} - \frac{8}{9}\ln z + \frac{8}{27} + \frac{4}{9}x \qquad (2.29)\\
& -\frac{2}{9}(2+x)|1-x|^{1/2}\begin{cases}\left(\ln\left|\frac{\sqrt{1-x}+1}{\sqrt{1-x}-1}\right| - i\pi\right), & \text{for } x \equiv \frac{4z^2}{\hat{s}} < 1 \\ 2\arctan\frac{1}{\sqrt{x-1}}, & \text{for } x \equiv \frac{4z^2}{\hat{s}} > 1,\end{cases} \\
h(0,\hat{s}) &= \frac{8}{27} - \frac{8}{9}\ln\frac{m_\mathrm{b}}{\mu} - \frac{4}{9}\ln\hat{s} + \frac{4}{9}i\pi. \qquad (2.30)\\
f(z) &= 1 - 8z^2 + 8z^6 - z^8 - 24z^4\ln z, \qquad (2.31)\\
\kappa(z) &= 1 - \frac{2\alpha_s(\mu)}{3\pi}\left[(\pi^2 - \frac{31}{4})(1-z)^2 + \frac{3}{2}\right] \qquad (2.32)\\
\tilde{\eta}(\hat{s}) &= 1 + \frac{\alpha_s(\mu)}{\pi}\,\omega(\hat{s}) \qquad (2.33)
\end{aligned}
$$

with

$$
\begin{aligned}
\omega(\hat{s}) &= -\frac{2}{9}\pi^2 - \frac{4}{3}\mathrm{Li}_2(s) - \frac{2}{3}\ln s\ln(1-s) - \frac{5+4s}{3(1+2s)}\ln(1-s) - \\
& \frac{2s(1+s)(1-2s)}{3(1-s)^2(1+2s)}\ln s + \frac{5+9s-6s^2}{6(1-s)(1+2s)}.
\end{aligned} \tag{2.34}
$$

Here $f(z)$ and $\kappa(z)$ are the phase-space factor and the single gluon QCD correction to the $b \to c e \bar{\nu}$ decay [26, 27] respectively. $\tilde{\eta}$ on the other hand represents single gluon corrections to the matrix element of Q_9 with $m_\mathrm{s} = 0$ [12, 28]. For consistency reasons this correction should only multiply the leading logarithmic term in P_0^{NDR}.

In the HV scheme the one-loop matrix elements are different and one finds an additional explicit contribution to (2.28) given by

$$
-\xi^{HV}\frac{4}{9}\left(3C_1^{(0)} + C_2^{(0)} - C_3^{(0)} - 3C_4^{(0)}\right). \tag{2.35}
$$

However $\widetilde{C}_9^{NDR}$ has to be replaced by $\widetilde{C}_9^{HV}$ given in (2.10) and (2.22) and consequently $\widetilde{C}_9^{eff}$ is the same in both schemes.

The first term in the function $h(z, \hat{s})$ in (2.29) represents the leading μ-dependence in the matrix elements. It is cancelled by the μ-dependence present in the leading logarithm in $\widetilde{C}_9$. The μ-dependence present in the coefficients of the other operators can only be cancelled by going to still higher order in the renormalization group improved perturbation theory. To this end the matrix elements of four-quark operators should be evaluated at two-loop level. Also certain unknown three-loop anomalous dimensions should be included in the evaluation of C_7^{eff} and C_9 [18, 19]. Certainly this is beyond the scope of the present paper and we will only investigate the left-over μ-dependence in sect. 4.

The fact that the coefficient C_9 should include next-to-leading logarithms and the other coefficients should be calculated in the leading logarithmic approximation is easy to understand. There is a large logarithm in C_9 represented by $1/\alpha_s$ in P_0 in (2.11). Consequently the renormalization group improved perturbation theory for C_9 has the structure $\mathcal{O}(1/\alpha_s) + \mathcal{O}(1) + \mathcal{O}(\alpha_s) + \ldots$ whereas the corresponding series for the remaining coefficients is $\mathcal{O}(1) + \mathcal{O}(\alpha_s) + \ldots$. Therefore in order to find the next-to-leading $\mathcal{O}(1)$ term, the full two-loop renormalization group analysis for the operators in (2.1) has to be performed in order to find C_9, but the coefficients of the remaining operators should be taken in the leading logarithmic approximation. This is gratifying because the coefficient of the magnetic operator Q_7 is known only in the leading logarithmic approximation. Q_7 does not mix with Q_9 and has no impact on the coefficients C_1–C_6. Consequently the necessary two-loop renormalization group analysis of C_9 can be performed independently of the presence of the magnetic operators, which was also the case of the decay $K_L \to \pi^0 e^+ e^-$ presented in ref. [19].

Let us finally compare our main formulae (2.27)–(2.35) with the ones given in the literature:

i) The general expression (2.27) with $\kappa(z) = 1$ is due to Grinstein et. al. [2] who in their approximate leading order renormalization group analysis kept only the operators $Q_1, Q_2, Q_7, Q_9, Q_{10}$.

ii) Inserting $C_i^{(0)}$ and $\widetilde{C}_9^{NDR}$ in (2.2)and (2.8) into (2.28) we find an analytic expression for $\widetilde{C}_9^{eff}$ which agrees with a recent independent calculation of Misiak [12].

iii) The sign of $i\pi$ in (2.29) differs from the one given in [2] and [11] but agrees with [12] and also with the work of Fleischer [29].

iv) The "ξ-term" given in (2.35) contains in the HV scheme also contributions from the operators Q_3 and Q_4, which are however negligible. The discussion of the "ξ-term" in refs. [2] and [11] does not apply then to the HV scheme.

3 Technical Details

3.1 Wilson Coefficients

In order to calculate the coefficient C_9 including next-to-leading order corrections we have to perform in principle a two-loop renormalization group analysis for the full set of operators given in (2.1). However, Q_{10} is not renormalized and the dimension five operators Q_7 and Q_8 have no impact on C_9. Consequently only a set of seven operators, Q_{1-6} and Q_9, has to be considered. This is precisely the case of the decay $K_L \to \pi^0 e^+ e^-$ considered in [19] except

for an appropriate change of quark flavours and the fact that now $\mu \approx \mathcal{O}(m_{\rm b})$ instead of $\mu \approx \mathcal{O}(1\,{\rm GeV})$ should be considered. Because our detailed NLO analysis of $K_L \to \pi^0 e^+ e^-$ has already been published we will only discuss very briefly an analogous calculation of $B \to X_s e^+ e^-$, referring the interested reader to [19]. We should stress that Misiak [11, 12] used different conventions for the evanescent operators than used in [19] and here. The agreement on $\widetilde{C}_9^{eff}$ is therefore particularly satisfying.

Integrating out simultaneously W, Z and t we construct first the effective Hamiltonian for $\Delta B = 1$ transitions relevant for $b \to s e^+ e^-$ with the operators normalized at $\mu = M_{\rm W}$. Dropping the operators Q_7, Q_8 and Q_{10} for the reasons stated above and using the unitarity of the CKM matrix we find

$$\begin{aligned}\mathcal{H}_{eff}(\Delta B = 1) &= -\frac{G_F}{\sqrt{2}} V_{ts}^* V_{tb} \left[\sum_{i=1}^{6} C_i(M_{\rm W}) Q_i + C_9'(M_{\rm W}) Q_9' \right] \\ &\quad + \frac{G_F}{\sqrt{2}} V_{us}^* V_{ub} \left[C_1(M_{\rm W}) (Q_1^{(u)} - Q_1) + C_2(M_{\rm W}) (Q_2^{(u)} - Q_2) \right]. \end{aligned} \tag{3.1}$$

Here $Q_{1,2}^{(u)}$ are obtained from $Q_{1,2}$ through the replacement $c \to u$. In order to make all the elements of the anomalous dimension matrix be of the same order in α_s, we have appropriately rescaled C_9 and Q_9:

$$Q_9' = \frac{\alpha}{\alpha_s(\mu)} Q_9, \qquad C_9'(\mu) = \frac{\alpha_s(\mu)}{\alpha} C_9(\mu) \tag{3.2}$$

Note that because of GIM cancellation there are no penguin contributions in the term proportional to $V_{us}^* V_{ub}$. They would appear only at scales $\mu < m_{\rm c}$ as was the case in $K_L \to \pi^0 e^+ e^-$. Since $|V_{us}^* V_{ub} / V_{ts}^* V_{tb}| < 0.02$ we will drop the second term in what follows.

The initial conditions at $\mu = M_{\rm W}$ for the coefficients C_1–C_6 in NDR and HV schemes have been given in sect. 2.4 and in the appendix A of ref. [19] respectively. Here it suffices to give only the initial condition for the coefficient C_9' (denoted by C_{7V}' in [19]) which reads:

$$C_9'(M_{\rm W}) = \frac{\alpha_s(M_{\rm W})}{2\pi} \left[\frac{Y(x_t)}{\sin^2 \Theta_W} - 4 Z(x_t) + \frac{4}{9}(1+\xi) \right], \tag{3.3}$$

where ξ has been defined in (2.23). The x_t dependence originates in box diagrams and in the γ- and Z-penguin diagrams [30].

With

$$\vec{C}^T \equiv (C_1, \ldots, C_6, C_9') \tag{3.4}$$

one can calculate the coefficients $C_i(\mu)$ by using the evolution operator $\hat{U}_5(\mu, M_{\rm W})$ relevant for an effective theory with $f = 5$ flavours:

$$\vec{C}(\mu) = \hat{U}_5(\mu, M_{\rm W}) \vec{C}(M_{\rm W}). \tag{3.5}$$

An explicit expression for $\hat{U}_5$ is given in sect. 2 of [19] where also the relevant expressions for one- and two-loop anomalous dimensions can be found. One only has to set $f = 5$, $u = 2$ and $d = 3$ in the formulae given in [19].

Using (3.5) and rescaling back the operator Q_9 we find at $\mu \approx \mathcal{O}(m_{\rm b})$

$$\mathcal{H}_{eff}(\Delta B = 1) = -\frac{G_F}{\sqrt{2}} V_{ts}^* V_{tb} \left[\sum_{i=1}^{6} C_i(\mu) Q_i + C_9(\mu) Q_9 \right] \tag{3.6}$$

with the coefficient $C_9(\mu)$ given in (2.10) and (2.22) for NDR and HV schemes respectively. The result for HV can either be found directly using (3.5) or by using the relation

$$\vec{C}^{HV}(\mu) = \left(\hat{1} - \frac{\alpha_s(\mu)}{4\pi}\Delta\hat{r}^T\right)\vec{C}^{NDR}(\mu) \tag{3.7}$$

with the matrix $\Delta\hat{r}$ given in appendix A of ref. [19].

3.2 One-Loop Matrix Elements

The operators Q_7 and Q_{10} contribute at this level of accuracy only through tree level matrix elements. Q_8 contributes only through the renormalization of Q_7 and its impact is only felt in $C_7^{(0)eff}$. The four-quark operators Q_{1-6}, contribute at one-loop level through the diagrams in fig. 1 where "$\otimes\otimes$" denotes the operator insertion. Finally at next-to-leading level $\mathcal{O}(\alpha_s)$ corrections to the matrix element $\langle Q_9\rangle$ have to be calculated.

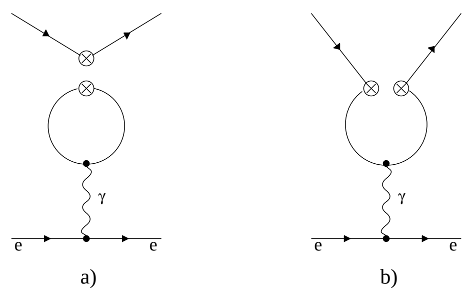

Figure 1: The two possibilites for insertion of a four-quark operator into a penguin diagram.

Let us begin with $\langle Q_{1-6}\rangle$. As usual two types of insertions of the operators into the penguin diagrams have to be considered. As already discussed in ref. [31] the appearance of a closed fermion loop in fig. 1a does not pose any problems in the NDR scheme because nowhere in the calculation one has to evaluate $\mathrm{Tr}[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\lambda\gamma_5]$. The diagrams in fig. 1 have been evaluated for the operators Q_1 and Q_2 by Grinstein et. al. [2] and by Misiak [11] for the full set Q_1–Q_6. These calculations have been done in the NDR scheme. Calculating these diagrams in the NDR and HV schemes we find

$$\begin{aligned}
\langle Q_1\rangle &= \frac{\alpha}{2\pi}\left(3h(z,\hat{s}) - \frac{4}{3}\xi\right)\langle Q_9\rangle_0 \\
\langle Q_2\rangle &= \frac{\alpha}{2\pi}\left(h(z,\hat{s}) - \frac{4}{9}\xi\right)\langle Q_9\rangle_0 \\
\langle Q_3\rangle &= \frac{\alpha}{2\pi}\left(3h(z,\hat{s}) - 2h(1,\hat{s}) - \frac{1}{2}h(0,\hat{s}) + \frac{2}{3} + \frac{4}{9}\xi\right)\langle Q_9\rangle_0 \\
\langle Q_4\rangle &= \frac{\alpha}{2\pi}\left(h(z,\hat{s}) - 2h(1,\hat{s}) - \frac{3}{2}h(0,\hat{s}) + \frac{2}{9} + \frac{4}{3}\xi\right)\langle Q_9\rangle_0 \\
\langle Q_5\rangle &= \frac{\alpha}{2\pi}\left(3h(z,\hat{s}) - \frac{3}{2}h(1,\hat{s}) + \frac{2}{3}\right)\langle Q_9\rangle_0 \\
\langle Q_6\rangle &= \frac{\alpha}{2\pi}\left(h(z,\hat{s}) - \frac{1}{2}h(1,\hat{s}) + \frac{2}{9}\right)\langle Q_9\rangle_0
\end{aligned} \tag{3.8}$$

with ξ defined in (2.23), $\langle Q_9\rangle_0$ denoting the tree level matrix element of Q_9 and

$$h(z,\hat{s}) = \frac{2}{3}G(z,\hat{s}) - \frac{4}{9} - \frac{8}{9}\ln\frac{m_{\rm b}}{\mu}. \tag{3.9}$$

Here

$$G(z,\hat{s}) = -4\int_0^1 dx\, x(1-x)\ln\Big(z^2 - \hat{s}x(1-x)\Big) \tag{3.10}$$

with z and $\hat{s}$ defined in (2.26).

A few remarks should be made:

- $h(z,\hat{s})$, $h(1,\hat{s})$ and $h(0,\hat{s})$ correspond to internal c, b and massless (u,d,s) quarks in fig. 1 respectively.
- The contributions of (u,d,s) to diagram 1a) cancel each other and consequently $h(0,\hat{s})$ represents the contribution of the internal strange quark in diagram 1b).
- We note that $\langle Q_5\rangle$ and $\langle Q_6\rangle$ matrix elements do not contain the ξ-term. We should however stress that generally it is certainly possible to find schemes in which $\langle Q_5\rangle$ and $\langle Q_6\rangle$ matrix elements can differ from the ones given in (3.8). Similarly we have no argument that in schemes different from NDR and HV the matrix elements are found simply by changing the value of ξ in the formulae given above. It could be that the changes are more involved. Consequently the discussions of the ξ-term presented in [2] and [11] are not generally valid.

The one gluon correction to the matrix element of Q_9, $\tilde{\eta}(\hat{s})$, can be inferred from [28] as has been noticed by Misiak in [12]. In [28] a left-handed current has been considered. Thus we rewrite the vector current as a sum of left- and right-handed currents. Neglecting the electron masses these two contributions do not interfere. Charge conjugation transforms the right-handed current into a left-handed one. Since $\hat{s}$ is invariant under this transformation both currents lead to the same invariant mass spectrum. Therefore we can write

$$\omega(\hat{s}) = \frac{-2}{(1-\hat{s})^2(1+2\hat{s})}\int_{\hat{s}}^1 dx\tilde{F}_1(x,\hat{s}) \tag{3.11}$$

with $\tilde{F}_1(x,\hat{s})$ defined explicitly in eq. (3.9) of [28]. Calculating the integral we arrive at the result given in (2.34) which furthermore agrees with Misiak [32].

4 Numerical Analysis

In our numerical analysis we will use

$$\alpha_s(\mu) = \frac{4\pi}{\beta_0\ln(\mu^2/\Lambda^2_{\overline{\rm MS}})}\left[1 - \frac{\beta_1}{\beta_0^2}\frac{\ln\ln(\mu^2/\Lambda^2_{\overline{\rm MS}})}{\ln(\mu^2/\Lambda^2_{\overline{\rm MS}})}\right] \tag{4.12}$$

with $\beta_0 = 23/3$ and $\beta_1 = 116/3$ as appropriate for five flavours. We also take $\Lambda_{\overline{\rm MS}} = (225 \pm 85)$ MeV corresponding to $\alpha_s(M_{\rm Z}) = 0.117 \pm 0.007$. For the remaining parameters we take

$$
\begin{aligned}
\alpha &= 1/129, & m_{\rm c} &= 1.4\,\text{GeV}, \\
\sin^2\theta_W &= 0.23, & m_{\rm b} &= 4.8\,\text{GeV}, \\
|V_{ts}/V_{cb}| &= 1, & M_{\rm W} &= 80.0\,\text{GeV}.
\end{aligned}
\tag{4.13}
$$

In table 1 we show the constant P_0 in (2.11) for different μ and $\Lambda_{\overline{\rm MS}}$, in the leading order corresponding to the first term in (2.11) and for the NDR and HV schemes as given by (2.11) and (2.22) respectively. In table 2 we show the corresponding values for $\widetilde{C}_9(\mu)$. To this end we set $m_{\rm t} = 170\,\text{GeV}$.

	$\Lambda_{\overline{\rm MS}} = 0.140\,\text{GeV}$			$\Lambda_{\overline{\rm MS}} = 0.225\,\text{GeV}$			$\Lambda_{\overline{\rm MS}} = 0.310\,\text{GeV}$		
$\mu[\,\text{GeV}]$	LO	NDR	HV	LO	NDR	HV	LO	NDR	HV
2.5	2.052	2.927	2.796	1.932	2.845	2.758	1.834	2.774	2.726
5.0	1.851	2.623	2.402	1.787	2.589	2.394	1.735	2.560	2.387
7.5	1.673	2.389	2.125	1.630	2.371	2.126	1.596	2.356	2.126
10.0	1.524	2.202	1.910	1.493	2.192	1.915	1.468	2.183	1.919

Table 1: The coefficient P_0 of $\widetilde{C}_9$ for various values of $\Lambda_{\overline{\rm MS}}$ and μ.

	$\Lambda_{\overline{\rm MS}} = 0.140\,\text{GeV}$			$\Lambda_{\overline{\rm MS}} = 0.225\,\text{GeV}$			$\Lambda_{\overline{\rm MS}} = 0.310\,\text{GeV}$		
$\mu[\,\text{GeV}]$	LO	NDR	HV	LO	NDR	HV	LO	NDR	HV
2.5	2.052	4.495	4.364	1.932	4.413	4.326	1.834	4.341	4.293
5.0	1.851	4.193	3.972	1.787	4.159	3.963	1.735	4.130	3.956
7.5	1.673	3.960	3.696	1.630	3.942	3.696	1.596	3.926	3.697
10.0	1.524	3.774	3.482	1.493	3.763	3.486	1.468	3.754	3.490

Table 2: Wilson coefficient $\widetilde{C}_9$ for $m_{\rm t} = 170\,\text{GeV}$ and various values of $\Lambda_{\overline{\rm MS}}$ and μ.

We observe:

- The NLO corrections to P_0 enhance this constant relatively to the LO result by roughly 45% and 35% in the NDR and HV schemes respectively. This enhancement is analogous to the one found in the case of $K_L \to \pi^0 e^+ e^-$.

- In calculating P_0 in the LO we have used $\alpha_s(\mu)$ at one-loop level. Had we used the two-loop expression for $\alpha_s(\mu)$ we would find for $\mu = 5\,\text{GeV}$ and $\Lambda_{\overline{\rm MS}} = 225\,\text{MeV}$ the value $P_0^{LO} \approx 1.98$. Consequently the NLO corrections would have smaller impact. Ref. [2] including the next-to-leading term 4/9 would find P_0 values roughly 20% smaller than P_0^{NDR} given in tab. 1.

- It is tempting to compare P_0 in table 1 with that found in the absence of QCD corrections. In the limit $\alpha_s \to 0$ we find $P_0^{NDR} = 8/9 \ln(M_{\rm W}/\mu) + 4/9$ and $P_0^{HV} = 8/9 \ln(M_{\rm W}/\mu)$ which for $\mu = 5\,\text{GeV}$ give $P_0^{NDR} = 2.91$ and $P_0^{HV} = 2.46$. Comparing these values with table 1 we conclude that the QCD suppression of P_0 present

in the leading order approximation is considerably weakened in the NDR treatment of γ_5 after the inclusion of NLO corrections. It is essentially removed for $\mu > 5\,\mathrm{GeV}$ in the HV scheme.

- The NLO corrections to $\widetilde{C}_9$ which include also the m_t-dependent contributions are large as seen in table 2. The results in HV and NDR schemes are by more than a factor of two larger than the leading order result $\widetilde{C}_9 = P_0^{LO}$ which consistently should not include m_t-contributions. This demonstrates very clearly the necessity of NLO calculation which allow a consistent inclusion of the important m_t-contributions. For the same set of parameters the authors of ref. [2] would find $\widetilde{C}_9$ to be smaller than $\widetilde{C}_9^{NDR}$ by 10–15%.

- The μ and $\Lambda_{\overline{MS}}$ dependences of $\widetilde{C}_9$ are quite weak. We also find that the m_t dependence of $\widetilde{C}_9$ is rather weak. Varying m_t between 150 GeV and 190 GeV changes $\widetilde{C}_9$ by at most 10%. This weak m_t dependence of $\widetilde{C}_9$ originates in the partial cancellation of m_t dependences between $Y(x_t)$ and $Z(x_t)$ in (2.10) as already seen in the case of $K_L \to \pi^0 e^+ e^-$. Finally, the difference between $\widetilde{C}_9^{NDR}$ and $\widetilde{C}_9^{HV}$ is small and amounts to roughly 5%.

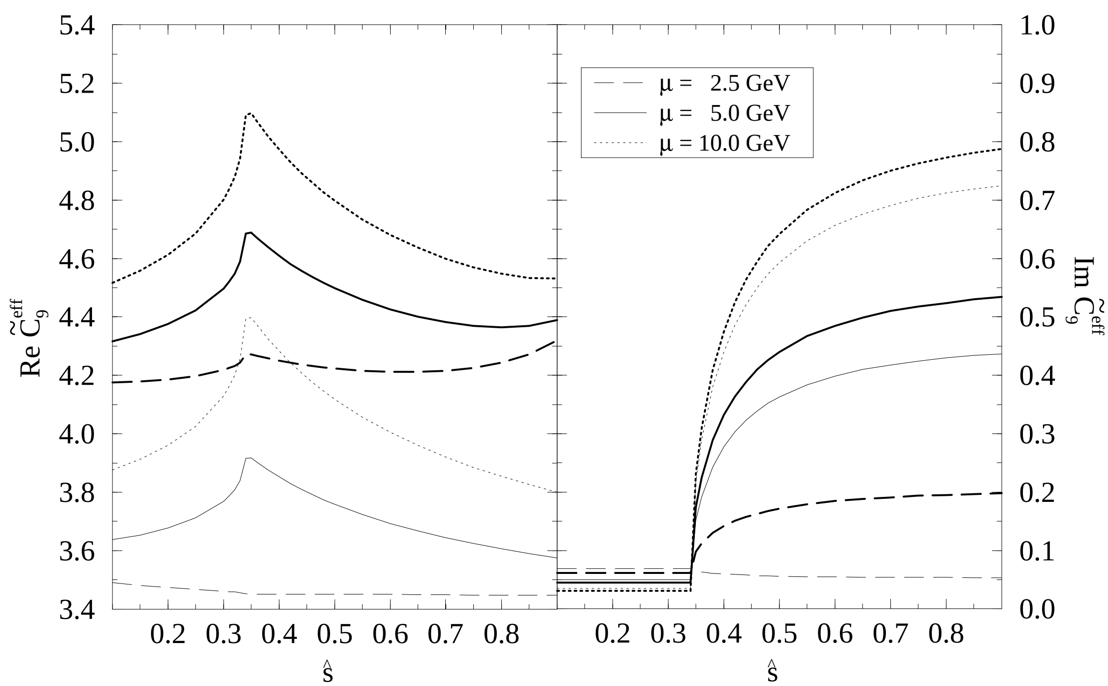

Figure 2: Comparison of the Wilson coefficient $\widetilde{C}_9^{eff}$ as a function of $\hat{s}$ for $m_t = 170\,\mathrm{GeV}$, $\Lambda_{\overline{MS}} = 0.225\,\mathrm{GeV}$ and different values of μ in leading order (thin lines) and next-to-leading order (thick lines) accuracy. Note the different scales for the real and imaginary parts!

In fig. 2 we show $\widetilde{C}_9^{eff}$ of (2.28) as a function of $\hat{s}$ for $m_t = 170\,\mathrm{GeV}$, $\Lambda_{\overline{MS}} = 225\,\mathrm{MeV}$ and $2.5 \le \mu \le 10\,\mathrm{GeV}$. In order to see the importance of the term resulting from the one-loop matrix elements one should compare these results with the $\hat{s}$-independent values of $\widetilde{C}_9$. We should also remember that the NLO corrections to P_0 calculated here shift $\widetilde{C}_9^{eff}$ for $\mu = 5.0\,\mathrm{GeV}$ by $\Delta\widetilde{C}_9^{NDR} \approx 0.8$ and $\Delta\widetilde{C}_9^{HV} \approx 0.6$ with similar results for other μ. In order

to show this effect more explicitly we also plot in fig. 2 a "leading order" result obtained by using only the leading term in (2.11) with α_s at the one-loop level but keeping otherwise all explicit NLO terms in (2.10) and the contributions from one-loop matrix elements given in (2.28). It should be stressed that roughly 50% of the difference between the "thick" and "thin" lines in fig. 2 is due to the term 4/9 in (3.3) which in the NDR scheme enters the NLO terms in P_0 but in the HV scheme is present in the one-loop matrix elements. We have left it out in the "thin" lines in fig. 2 in order to show its importance. The calculation of NLO corrections to P_0 allows a consistent inclusion of this term which contributes positively to $\widetilde{C}_9^{eff}$. Additional enhancement comes from using the two-loop renormalization group analysis for $\widetilde{C}_9$ and α_s at the two-loop level. In fig. 2 we also note that $\mathrm{Re}\widetilde{C}_9^{eff} \gg \mathrm{Im}\widetilde{C}_9^{eff}$. The pronounced peak for $\hat{s} = 4m_\mathrm{c}^2/m_\mathrm{b}^2 = 0.34$ is related to the behaviour of $h(z,\hat{s})$ in (2.29). This peak essentially disappears for $\mu = 2.5\,\mathrm{GeV}$ because of the accidential cancellation $3C_1^{(0)} + C_2^{(0)} \approx 0$ in the dominant term multiplying $h(z,\hat{s})$. The authors of ref. [2] would find $\mathrm{Re}\,\widetilde{C}_9^{eff}$ by about 15% below our values. In the absence of QCD corrections, $h(z,\hat{s})$ in (2.28) is multiplied by $C_2^{(0)} = 1$ and consequently there is no accidental suppression of this term as in the QCD case. Since in addition for $\alpha_s \to 0$ P_0^{NDR} is slightly enhanced over the values given in table 1, we find $\widetilde{C}_9^{eff}$ in the absence of QCD corrections to be substantially larger than the result given in fig. 2. For instance, $\mathrm{Re}\,\widetilde{C}_9^{eff}$ varies between 5.2 and 6.3 for $0.1 \le \hat{s} \le 0.9$. The complete result for $R(\hat{s})$ in this case is shown in fig. 5 at the end of this section.

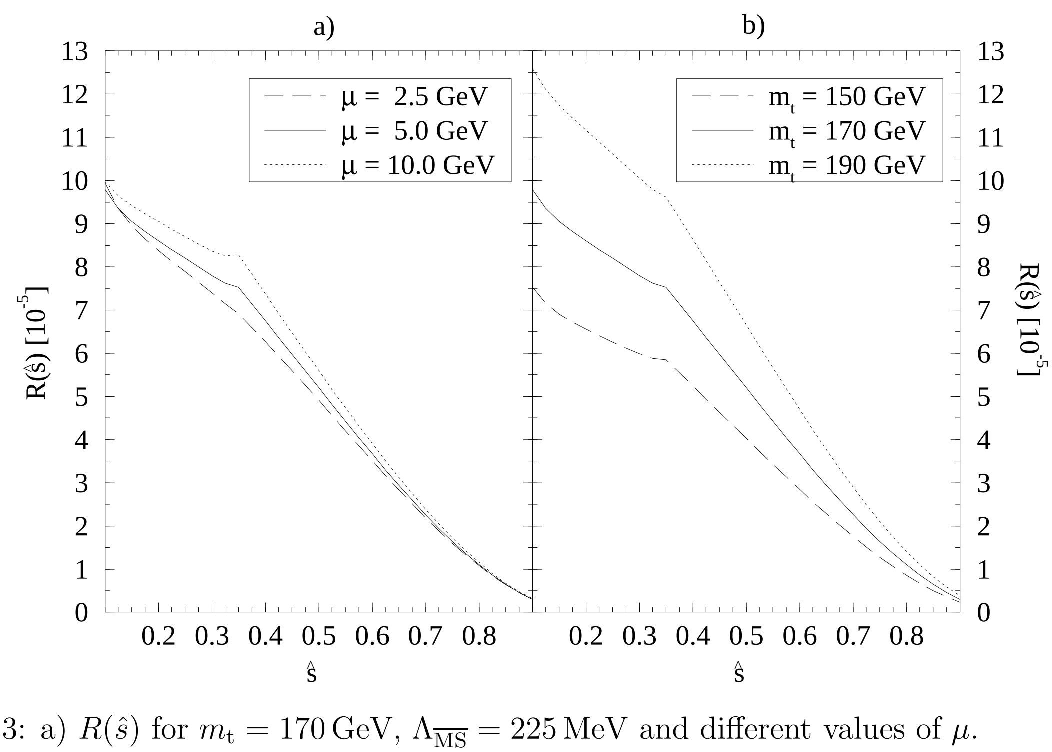

Figure 3: a) $R(\hat{s})$ for $m_\mathrm{t} = 170\,\mathrm{GeV}$, $\Lambda_{\overline{\mathrm{MS}}} = 225\,\mathrm{MeV}$ and different values of μ.
b) $R(\hat{s})$ for $\mu = 5\,\mathrm{GeV}$, $\Lambda_{\overline{\mathrm{MS}}} = 225\,\mathrm{MeV}$ and various values of m_t.

We next present a numerical analysis of (2.27). In doing this we keep in mind that for $\hat{s} \approx m_\psi^2/m_\mathrm{b}^2$, $\hat{s} \approx m_{\psi'}^2/m_\mathrm{b}^2$ etc. the spectator model cannot be the full story and additional long distance contributions discussed in refs. [14–16] have to be taken into account in a phenomenological analysis. Similarly we do not include $1/m_\mathrm{b}^2$ corrections calculated in [13]

which typically enhance the differential rate by about 10%.

In fig. 3a) we show $R(\hat{s})$ for $m_t = 170\,\text{GeV}$, $\Lambda_{\overline{\text{MS}}} = 225\,\text{MeV}$ and different values of μ. In fig. 3b) we set $\mu = 5\,\text{GeV}$ and vary m_t from 150 GeV to 190 GeV. The remaining μ dependence is rather weak and amounts to at most $\pm 8\%$ in the full range of parameters considered. The m_t dependence of $R(\hat{s})$ is sizeable. Varying m_t between 150 GeV and 190 GeV changes $R(\hat{s})$ by typically 60–65% which in this range of m_t corresponds to $R(\hat{s}) \sim m_t^2$. It is easy to verify that this strong m_t dependence originates in the coefficient $\widetilde{C}_{10}$ given in (2.8) as already stressed by several authors in the past [1–8].

We do not show the $\Lambda_{\overline{\text{MS}}}$ dependence as it is very weak. Typically, changing $\Lambda_{\overline{\text{MS}}}$ from 140 MeV to 310 MeV decreases $R(\hat{s})$ by about 5%.

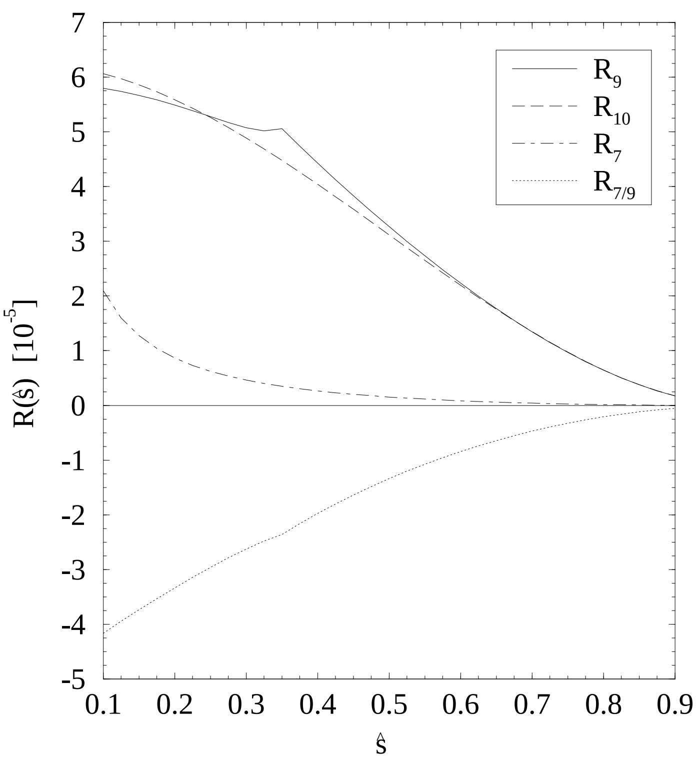

Figure 4: Comparison of the four different contributions to $R(\hat{s})$ according to eqn. (2.27).

$R(\hat{s})$ is governed by three coefficients, $\widetilde{C}_9^{eff}$, $\widetilde{C}_{10}$ and $C_7^{(0)eff}$. It is of interest to investigate the importance of various contributions. To this end we set $\Lambda_{\overline{\text{MS}}} = 225\,\text{GeV}$, $m_t = 170\,\text{GeV}$ and $\mu = 5\,\text{GeV}$. In fig. 4 we show $R(\hat{s})$ keeping only $\widetilde{C}_9^{eff}$, $\widetilde{C}_{10}$, $C_7^{(0)eff}$ and the $C_7^{(0)eff}$–$\widetilde{C}_9^{eff}$ interference term, respectively. Denoting these contributions by R_9, R_{10}, R_7 and $R_{7/9}$ we observe that the term R_7 plays only a minor role in $R(\hat{s})$. On the other hand the presence of $C_7^{(0)eff}$ cannot be ignored because the interference term $R_{7/9}$ is significant. In fact the presence of this large interference term could be used to measure experimentally the relative sign of $C_7^{(0)eff}$ and $\text{Re}\,\widetilde{C}_9^{eff}$ [2, 4, 5, 7, 8] which as seen in fig. 4 is negative in the Standard Model. However, the most important contributions are R_9 and R_{10} in the full range of $\hat{s}$ considered. For $m_t \approx 170\,\text{GeV}$ these two contributions are roughly of the same size. Due to a strong m_t dependence of R_{10}, this contribution dominates for higher values of m_t and is less important than R_9 for $m_t < 170\,\text{GeV}$.

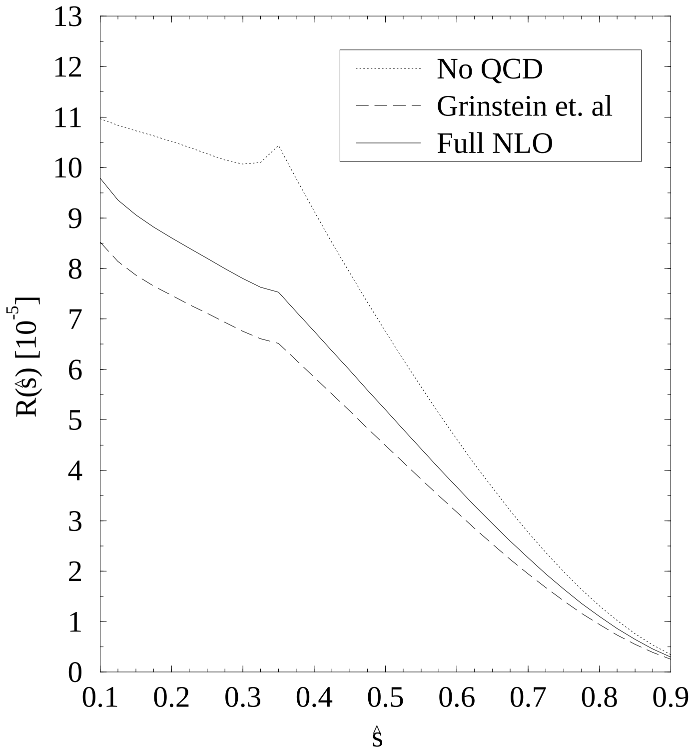

Figure 5: $R(\hat{s})$ for $m_\mathrm{t} = 170\,\mathrm{GeV}$, $\Lambda_{\overline{\mathrm{MS}}} = 225\,\mathrm{MeV}$ and $\mu = 5\,\mathrm{GeV}$.

Next, in fig. 5 we show $R(\hat{s})$ for $\mu = 5\,\mathrm{GeV}$, $m_\mathrm{t} = 170\,\mathrm{GeV}$ and $\Lambda_{\overline{\mathrm{MS}}} = 225\,\mathrm{MeV}$ compared to the case of no QCD corrections and to the results Grinstein et. al. [2] would obtain for our set of parameters using their approximate leading order formulae.

Finally, we would like to address the question of the definition of m_t used here. In order to be able to analyze this question, one would have to calculate perturbative QCD corrections to the functions $Y(x_t)$ and $Z(x_t)$ and include also an additional order in the renormalization group improved perturbative calculation of P_0. The latter would require evaluation of three-loop anomalous dimension matrices, which in the near future nobody will attempt. In any case, we expect only a small correction to P_0. The uncertainty due to the choice of μ in $m_\mathrm{t}(\mu)$ can be substantial, as stressed in refs. [33,34], and may result in 20–30% uncertainties in the branching ratios. It can only be reduced if $\mathcal{O}(\alpha_s)$ corrections to $Y(x_t)$ and $Z(x_t)$ are included. For $K^+ \to \pi^+\nu\bar{\nu}$, $K_L \to \pi^0\nu\bar{\nu}$, $B \to \mu^+\mu^-$ and $B \to X_s\nu\bar{\nu}$ this has been done in refs. [33,34]. The inclusion of these corrections reduces the uncertainty in the corresponding branching ratios to a few percent. Fortunately, the result for the corrected function $Y(x_t)$ given in refs. [33,34] can be directly used here. The message of refs. [33,34] is the following: For $m_\mathrm{t} = \overline{m}_\mathrm{t}(m_\mathrm{t})$, the QCD corrections to $Y(x_t)$ and consequently to $\widetilde{C}_{10}$ are below 2%. Corresponding corrections to $Z(x_t)$ are not known. Fortunately, the m_t dependence of $\widetilde{C}_9$ is much weaker and the uncertainty due to the choice of μ in $m_\mathrm{t}(\mu)$ is small. On the basis of these arguments and the result of refs. [33,34] we believe that if $m_\mathrm{t} = \overline{m}_\mathrm{t}(m_\mathrm{t})$ is chosen, the additional short distance QCD corrections to $\mathrm{BR}(B \to X_s e^+e^-)$ should be small.

5 Summary

We have calculated the effective Hamiltonian relevant for the rare decay $B \to X_s e^+e^-$ beyond the leading logarithmic approximation. The main new result of this paper is the calculation of the Wilson coefficient of the operator $Q_9 = (\bar{s}b)_{V-A}(\bar{e}e)_V$ including next-to-leading logarithms in the NDR and HV renormalization schemes. A separate analytic expression for C_9 given in sect. 2 as opposed to C_9^{eff} given in [12] should be useful not only in $B \to X_s e^+e^-$ but also in $B \to K^* e^+e^-$ and other rare B-decays to which Q_9 contributes. Calculating $B \to X_s e^+e^-$ in the spectator model we confirm the very recent result for C_9^{eff} presented by Misiak in [12]. The cancellation of the scheme dependence in C_9^{eff} is shown explicitly in our paper.

The effect of the NLO corrections is to enhance $\mathrm{BR}(B \to X_s e^+e^-)$ so that its suppression found in the leading order analysis of ref. [2] is considerably weakened. This is seen in particular in fig. 5.

We have investigated the m_t, $\Lambda_{\overline{\mathrm{MS}}}$ and $\mu \approx \mathcal{O}(m_\mathrm{b})$ dependence of the "reduced" branching ratio $R(\hat{s})$. The dependences on $\Lambda_{\overline{\mathrm{MS}}}$ and μ are rather small, at most $\pm 8\%$ in the full range of parameters considered. The dependence on m_t is sizeable. In the range $150\,\mathrm{GeV} \le m_\mathrm{t} \le 190\,\mathrm{GeV}$ it is roughly parametrized by $R(\hat{s}) \sim m_\mathrm{t}^2$. For $m_\mathrm{t} = 170\,\mathrm{GeV}$, $\Lambda_{\overline{\mathrm{MS}}} = 225\,\mathrm{MeV}$, $\mu = 5\,\mathrm{GeV}$ and $0.1 \le \hat{s} \le 0.8$ we find

$$1.0 \cdot 10^{-5} \le R(\hat{s}) \le 9.8 \cdot 10^{-5}. \tag{5.14}$$

This result can be modified by non-perturbative $1/m_\mathrm{b}^2$ corrections and long distance contributions [14–16], which are however beyond the scope of this paper.

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